1.

The generating functional for the Yukawa theory is

$$Z[J,\eta,\bar\eta] = \int\mathcal{D}\varphi\int\mathcal{D}\psi\int\mathcal{D}\bar\psi\; e^{i\int d^4x\,\{\mathcal{L}+J\varphi+\bar\eta\psi+\bar\psi\eta\}}$$

with external sources $J, \eta, \bar\eta$.

We expand in powers of the coupling g in $\mathcal{L} \equiv \mathcal{L}_0 + g\bar\psi\psi\varphi$

$$Z[J,\eta,\bar\eta] = \int\mathcal{D}\varphi\int\mathcal{D}\psi\int\mathcal{D}\bar\psi\; e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\bar\eta\psi+\bar\psi\eta\}}\times$$
$$\times\left\{1+ig\int d^4w\;\bar\psi(w)\psi(w)\varphi(w)+O(g^2)\right\}$$

We can now rewrite the fields $\varphi, \psi, \bar\psi$ as derivatives w.r.t. external sources

$$\varphi(w)\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\bar\eta\psi+\bar\psi\eta\}} = \frac{\delta}{\delta iJ(w)}\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\dots\}}$$

$$\psi(w)\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\bar\eta\psi+\bar\psi\eta\}} = \frac{\delta}{\delta i\bar\eta(w)}\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\dots\}}$$

$$\bar\psi(w)\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\bar\eta\psi+\bar\psi\eta\}} = \frac{\delta}{\delta(-i\eta(w))}\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\dots\}}$$

Thus one obtains

$$Z[J,\eta,\bar\eta] = \int\mathcal{D}\varphi\int\mathcal{D}\psi\int\mathcal{D}\bar\psi\left\{1+ig\int d^4w\,\frac{\delta^3}{\delta(-i\eta(w))\,\delta(i\bar\eta(w))\,\delta iJ(w)}+O(g^2)\right\}\times$$
$$\times\, e^{i\int d^4x\,\{\mathcal{L}_0+J\varphi+\bar\eta\psi+\bar\psi\eta\}}$$

The expansion in g can be brought out of the integral and resummed to obtain

$$Z[J,\eta,\bar\eta] = e^{ig\int d^4w \frac{\delta^3}{\delta(-i\eta)\,\delta i\bar\eta\,\delta iJ}} \int \mathcal{D}\varphi \int \mathcal{D}\psi \int \mathcal{D}\bar\psi \; e^{i\int d^4x \{\mathcal{L}_0 + J\varphi + \bar\psi\eta + \bar\eta\psi\}}$$

The first exponential now acts on the path integral of the free theory in the presence of external sources. This integral can be done exactly. Here the free theories of the scalar field and the fermions simply factorize to give

$$Z[J,\eta,\bar\eta] = Z[J=\eta=\bar\eta=0, g=0]\; e^{ig\int d^4w \frac{\delta^3}{\delta(-i\eta)\,\delta i\bar\eta\,\delta iJ}} \; e^{-\frac{i}{2}\int d^4x \int d^4y\, J(x)\, D(x-y)\, J(y)}$$

$$\times\; e^{-i\int d^4x \int d^4y\, \bar\eta(x)\, S(x-y)\, \eta(y)}$$

The derivation of each free factor goes as follows

$$\int \mathcal{D}\varphi\, e^{i\int d^4x \{\mathcal{L}_0 + J\varphi\}} = \int \mathcal{D}\varphi\, e^{i\int d^4x \{-\frac{1}{2}\varphi(\partial^2+M^2)\varphi + J\varphi\}}$$

$$K \equiv -(\partial^2+M^2)$$

$$\frac{1}{2}\varphi K\varphi + J\varphi = \frac{1}{2}(\varphi' - JK^{-1})\,K\,(\varphi' - K^{-1}J) + J(\varphi' - K^{-1}J)$$

$$= \frac{1}{2}\varphi' K \varphi' - \frac{1}{2} J K^{-1} J$$

with K^{-1} solution of $\quad -(\partial_x^2 + M^2)\, D(x-y) = \delta^{(4)}(x-y)$

$$\int \mathcal{D}\varphi\, e^{i\int d^4x \{\mathcal{L}_0 + J\varphi\}} = Z[J=0]\; e^{-\frac{i}{2}\int d^4x \int d^4y\, J(x)\, D(x-y)\, J(y)}$$

Analogously for fermions

$$\int \mathcal{D}\psi \int \mathcal{D}\bar\psi \; e^{i\int d^4x \left\{ \bar\psi (i\not\partial - m)\psi + \bar\eta \psi + \bar\psi \eta \right\}}$$

$$K = i\not\partial - m$$

$$\bar\psi K \psi + \bar\eta\psi + \bar\psi\eta = \left(\bar\psi' - \bar\eta K^{-1}\right) K \left(\psi' - K^{-1}\eta\right) + \bar\eta\left(\psi' - K^{-1}\eta\right) + \left(\bar\psi - \bar\eta K^{-1}\right)\eta$$

$$= \bar\psi' K \psi' - \bar\eta K^{-1}\eta$$

Thus

$$\int \mathcal{D}\psi \int \mathcal{D}\bar\psi \; e^{i\int d^4x \left\{ \bar\psi (i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta \right\}} = \cancel{Z[\eta = \bar\eta = 0]\, e^{-i\int d^4x \int d^4y\, \bar\eta(x)}}$$

$$= Z[\eta = \bar\eta = 0]\; e^{-i\int d^4x \int d^4y\, \bar\eta(x)\, S(x-y)\, \eta(y)}$$

where $S(x-y)$ is solution of $(i\not\partial_x - m)\, S(x-y) = \delta^{(4)}(x-y)$

<u>NB</u> I did not explicitly ~~used~~ write the $i\varepsilon$ prescription. The subtraction $m \to m - i\varepsilon$ is there whenever working in Minkowski spacetime.

2a.

The action in Natural Units ($\hbar = c = 1$) is dimensionless

$[S] = 0$ We count dimensions as powers of energy – i.e. $[\text{mass}] = 1$

$$S = \int d^d x \left\{ \mathcal{L}_0 - g \bar{\psi} \psi \varphi \right\}$$

$S = \int d^d x\, \mathcal{L}$ implies $[\mathcal{L}] = d$

This provides the dimensions of the fields $\varphi, \psi, (\bar{\psi})$ using the derivative or the mass terms in $\mathcal{L}_0$

From $\partial_\mu \varphi \partial^\mu \varphi$ one obtains $d = 2[\partial_\mu] + 2[\varphi] = 2 + 2[\varphi]$

and $[\varphi] = \dfrac{d-2}{2}$

From $m \bar{\psi} \psi$ one obtains $d = [m] + 2[\psi] = 1 + 2[\psi]$

and $[\psi] = \dfrac{d-1}{2}$

The interaction term $g \bar{\psi} \psi \varphi$ thus provides the dimension of g

$$d = [g] + 2[\psi] + [\varphi] = [g] + 2\left(\frac{d-1}{2}\right) + \left(\frac{d-2}{2}\right)$$

and $[g] = d - 2\left(\dfrac{d-1}{2}\right) - \left(\dfrac{d-2}{2}\right) = 2 - \dfrac{d}{2} = \dfrac{4-d}{2}$

2b.

The theory is renormalizable when $[g] = 0$.
This happens for $d = 4$.

In this case the number of vertices V does not appear in the formula for the superficial degree of divergence D. For $d=4$

$$D = 4 - E_B - \frac{3}{2} E_F$$

D only depends on the number of external lines (fermionic and bosonic) in a given amplitude.
This implies that only a finite number of amplitude (not diagrams!) has $D \geq 0$, i.e. is superficially divergent.
~~For~~ The theory is then renormalizable.
Notice that for each of the divergent amplitudes there are infinitely many divergent Feynman diagrams.

NB When we talk about amplitude here we refer to a subclass of primitively divergent and 1PI (one-particle-irreducible) amplitudes.

$d < 4$ superrenormalizable $D = \ldots -V \ldots$

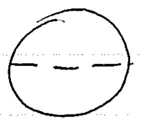
The theory has a finite number of divergent diagrams. ~~All amplitudes~~

$d > 4$ nonrenormalizable $D = \ldots +V \ldots$

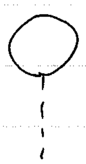
All amplitudes will diverge at a sufficiently high order in the loop expansion (i.e. sufficiently high number of vertices V or, equivalently, powers of the coupling g)

Example:

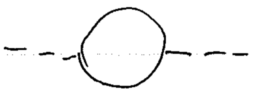
1PI diagrams in Yukawa theory in $d=4$ with $D \geq 0$



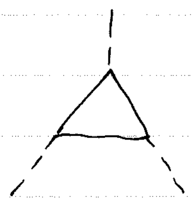
$D = 4$



$D = 3$

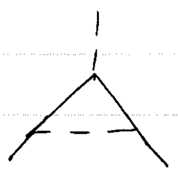


$D = 2$

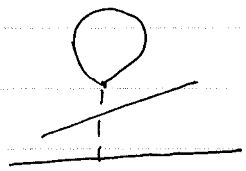


$D = 1$

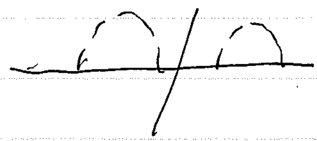
$D = 1$



$D = 0$



Not 1PI



Not 1PI

3a. $\psi' = e^{i\alpha\gamma_5}\psi$

$\overline{\psi} = \psi^{\dagger}\gamma^0$

$\overline{\psi}' = \psi'^{\dagger}\gamma^0 = \left(e^{i\alpha\gamma_5}\psi\right)^{\dagger}\gamma^0 = \psi^{\dagger}\left(e^{i\alpha\gamma_5}\right)^{\dagger}\gamma^0 = \psi^{\dagger}e^{-i\alpha\gamma_5^{\dagger}}\gamma^0$

$= \overline{\psi}\,\gamma^0 e^{-i\alpha\gamma_5}\gamma^0 = \overline{\psi}\,e^{i\alpha\gamma_5}$

We have used $\gamma_5^{\dagger} = \gamma_5$ and $\{\gamma_5, \gamma^0\} = 0$ $\quad \gamma^{0\,2} = \mathbb{1}$

$\gamma^0 e^{-i\alpha\gamma_5}\gamma^0 = \gamma^0\{1 - i\alpha\gamma_5 + O(\alpha^2)\}\gamma^0 = \gamma^{0\,2}\{1 + i\alpha\gamma_5 + O(\alpha^2)\}$

$= e^{i\alpha\gamma_5}$

3b. $\overline{\psi}' i\partial\!\!\!/ \psi' = \overline{\psi}\,e^{i\alpha\gamma_5}\,i\partial\!\!\!/\,e^{i\alpha\gamma_5}\psi = \overline{\psi}\,e^{i\alpha\gamma_5}e^{-i\alpha\gamma_5}\,i\partial\!\!\!/\psi$

$= \overline{\psi}\,i\partial\!\!\!/\psi$

$m\overline{\psi}'\psi' = m\overline{\psi}\,e^{i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi = m\overline{\psi}\,e^{2i\alpha\gamma_5}\psi \neq m\overline{\psi}\psi$

NB : ordering must be kept: $\overline{\psi}\,e^{2i\alpha\gamma_5}\psi = \overline{\psi}_{\alpha}\left(e^{2i\alpha\gamma_5}\right)_{\alpha\beta}\psi_{\beta}$

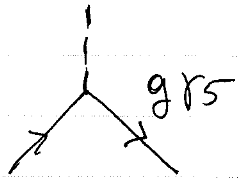
is a scalar.

Instead $e^{2i\alpha\gamma_5}\,\overline{\psi}\psi = \left(e^{2i\alpha\gamma_5}\right)_{\alpha\beta}\overline{\psi}_{\rho}\psi_{\rho}$

is a matrix 4×4 ! (~~as~~ ~~stated~~ in $d=4$)

We conclude that $\bar{\psi}\, i \not{\partial}\, \psi$ is invariant under the chiral transformation, while the mass term $m \bar{\psi} \psi$ is not.

4a. $\quad \varphi(p) \to \ell^-(p_1)\, \ell^+(p_2)$



$- - -$ scalar field φ (no polarization) spin $= 0$

$$A = g\, \bar{u}_r(p_1)\, \gamma_5\, v_s(p_2)$$

(follow the fermion arrow) keep ordering!

$$A^+ = g \left(\bar{u}_r(p_1)\, \gamma_5\, v_s(p_2) \right)^+ = g\, v_s^+(p_2)\, \gamma_5^+\, \bar{u}_r^+(p_1) =$$

$$= g\, \bar{v}_s(p_2)\, \gamma^0 \gamma_5 \left(u_r^+(p_1) \gamma^0 \right)^+ = g\, \bar{v}_s(p_2)\, \gamma^0 \gamma_5\, \gamma^0\, u_r(p_1)$$

$$= -g\, \bar{v}_s(p_2)\, \gamma_5\, u_r(p_1)$$

We have used $\gamma^{0+} = \gamma^0$, $\gamma^{0\,2} = \mathbb{1}$, $\{\gamma_5, \gamma^0\} = 0$

Notice $\bar{u}_r(p_1)\, \gamma_5\, v_s(p_2) = \bar{u}_r^{\alpha}(p_1)\, \gamma_5^{\alpha\beta}\, v_s^{\beta}(p_2)$ is a scalar

$\gamma_5\, \bar{u}_r(p_1)\, v_s(p_2) = \gamma_5^{\alpha\beta}\, \bar{u}_r^{\lambda}(p_1)\, v_s^{\lambda}(p_2)$ matrix!

The minus sign in A^+ will ensure positivity of A^+A.

$$X = \sum_{r,s=1,2} A^+ A = \sum_{r,s=1,2} (-g^2)\, \bar{v}_s(p_2)\, \gamma_5\, u_r(p_1)\, \bar{u}_r(p_1)\, \gamma_5\, v_s(p_2)$$

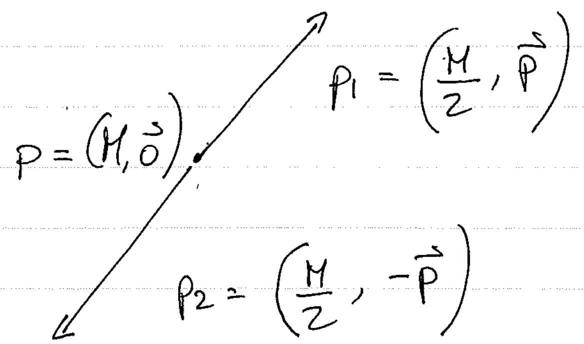
$$= -g^2 \, \mathrm{Tr}\left[\frac{\not p_2 - m}{2m}\gamma_5 \frac{\not p_1 + m}{2m}\gamma_5\right]$$

$$= -\frac{g^2}{4m^2} \, \mathrm{Tr}\left[(\not p_2 - m)(-\not p_1 + m)\right] \qquad \{\gamma_5, \gamma^\mu\} = 0$$

$$= \frac{g^2}{4m^2} \, \mathrm{Tr}\left[\not p_1 \not p_2 + m^2\right]$$ Notice the signs

$$= \frac{g^2}{m^2}\left(p_1 p_2 + m^2\right)$$

4b. Relativistic kinematics in CoM frame



p_1, p_2 obtained by imposing conservation of four-momentum
$p = p_1 + p_2$

$p_1 p_2$ can be obtained by squaring the relation $p = p_1 + p_2$

$$p^2 = (p_1 + p_2)^2 \implies M^2 = 2m^2 + 2 p_1 p_2 \implies p_1 p_2 = \frac{M^2}{2} - m^2$$

or directly by components $\quad p_1 p_2 = p_1^0 p_2^0 - \vec{p}_1 \cdot \vec{p}_2 = \frac{M^2}{4} + |\vec{p}|^2$

$$= \frac{M^2}{4} + \frac{M^2}{4} - m^2 = \frac{M^2}{2} - m^2$$

$$|\vec{p}|^2 = {p_1^0}^2 - m^2 = \frac{M^2}{4} - m^2$$

Inserting in X one obtains

$$X = \frac{g^2}{m^2}\left(\frac{M^2}{2} - \cancel{m^2} + \cancel{m^2}\right) = \frac{g^2 M^2}{2m^2}$$

which gives the decay rate

$$\Gamma = \frac{g^2}{8\pi} M \sqrt{1 - \frac{4m^2}{M^2}}$$

Check dimensions $\Gamma \sim$ energy $\sim \frac{1}{\text{lifetime}}$ (Natural Units)

For $M < 2m$ Γ becomes imaginary due to the phase space factor $\sqrt{1-\frac{4m^2}{M^2}}$.

This is when the total energy of the initial state is below threshold, i.e. not sufficient to produce the particles in the final state at rest.

The decay in $f^- f^+$ does not occur.

<u>NB</u> φ ~~can~~ may still decay to other final states, provided it is above threshold.